Algebraic Geometry II
Homework 2
Due Friday, January 30

(1) Show that there is a one-to-one correspondence between ring homomorphisms $R \to S$ and maps of affine schemes $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

(2) (a) Show that given a scheme X, maps from X to $\mathrm{Spec}(R)$ are in one-to-one correspondence with ring homomorphisms from R to $\Gamma(X, \mathcal{O}_X)$.
(b) Show that each scheme has a unique map to $\mathrm{Spec}(\mathbb{Z})$.
(c) Show that there is a one-to-one correspondence between regular functions on X and maps to $\mathbb{A}^1_{\mathbb{Z}}$. Use the Yoneda lemma to show that this property characterizes $\mathbb{A}^1_{\mathbb{Z}}$.

(3) Show that the only regular maps on $\mathbb{P}^n_k$ with $k = \overline{k}$ are the constants.

(4) Show that if R and S are graded rings then any graded homomorphism gives rise to a map $\mathrm{Proj}(S) \setminus V(S_{>0}) \to \mathrm{Proj}(R)$. Note that the converse is false.

(5) Let S be a graded ring generated in degree 1. Consider the subring S_{d*} (called the dth Veronese subring of S) generated by homogeneous elements of degrees divisible by d. Show that the natural map $\mathrm{Proj}(S) \to \mathrm{Proj}(S_{d*})$ is an isomorphism. Hint: Show that we can identify the principal open subsets of $\mathrm{Proj}(S_{d*})$ with open subsets of $\mathrm{Proj}(S)$ which cover it.

(6) Consider the scheme $\mathrm{Spec}(k[x, y]/(y^2, xy))$. What scheme corresponds to the quotient of this ring by its nilradical? Which stalks are reduced? Geometrically, we think of this as a variety with some fuzziness at a point.

(7) Show that if $k = \overline{k}$, there is a one-to-one correspondence between closed points of $\mathbb{P}(V)$ and one-dimensional quotients of V.

(8) Let P be a property of affine open subschemes of a scheme X which satisfies the following properties.
- If P holds for $\mathrm{Spec}(A)$, then it holds for $\mathrm{Spec}(A_f)$ for all $f \in A$.
- If $(f_1, \ldots, f_n) = A$, and P holds for all the $\mathrm{Spec}(A_{f_i})$ then P holds for $\mathrm{Spec}(A)$.

Show that if X has a cover by affine opens for which P holds, then every affine open of X has property P.