Algebraic Geometry II
Homework 3
Due Friday, February 6

(1) Show that a scheme is reduced if and only if all its stalks are reduced.

(2) Show that a *connected* scheme is integral if and only if all its stalks are integral domains.

(3) Show that an affine scheme $Spec(R)$ is disconnected if and only if $R \cong R_1 \times R_2$. with R_1 and R_2 nonzero rings.

(4) Show that a finite-dimensional reduced (i.e., no nilpotents) algebra over a field is a product of field extensions.

(5) Show that a finite morphism is quasi-finite.

(6) Show that a morphism being finite can be checked on a single open cover.

(7) We will now show how to use schemes to parametrize all Pythagorean triples. Consider the scheme $X = Spec(\mathbb{Q}[x, y]/(x^2 + y^2 - 1))$.
   (a) Show that the complement of the point $(1, 0)$ is isomorphic to $\mathbb{A}^1_{\mathbb{Q}}$ using stereographic projection.
   (b) Find all the maps from $Spec(\mathbb{Q})$ to X.
   (c) Describe all the rational solutions to the equation $x^2 + y^2 = 1$.
   (d) Describe all Pythagorean triples.

(8) Show that closed immersions are preserved under base change by affine morphisms, i.e., if $i : X \to Y$ is a closed immersion, then for all affine morphisms $f : Z \to Y$, the natural map $X \times_Y Z \to Z$ is also a closed immersion.

(9) Consider the natural map $Spec(\mathbb{Z}[\sqrt{2}]) \to Spec(\mathbb{Z})$. What scheme-theoretic fibers occur?

(10) What is the scheme-theoretic intersection of the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$ with the x-axis? What about the y-axis?