

Algebraic Geometry II  
Homework 5  
Due Friday, March 6

- (1) Let  $X \subset \mathbb{A}^3$  be the quadric cone  $V(xy - z^2)$ . Let  $D = V(x, z) \subset X$ . Show that  $\mathcal{O}_X(D)$  is not an invertible sheaf. Hint: think of the tangent space at the origin.
- (2) Show that the pushforward of a quasicoherent sheaf by a qcqs (quasi-compact quasi-separated) morphism is again quasicoherent.
- (3) Show that  $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$ . Note that since  $\text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ , we have an example of two smooth projective birational surfaces which are not isomorphic. Hint: Use the fact that the pullback of an invertible sheaf is invertible.
- (4) Show that a rational section of a line bundle gives a well-defined Weil divisor.
- (5) Consider  $\mathbb{P}^1 \times \mathbb{P}^1$  with its two projection maps  $\pi_1$  and  $\pi_2$ . Consider the invertible sheaf  $L = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ . Show that  $L$  is very ample. What is the dimension of the complete linear system of  $L$ ? What do you think the image of  $\mathbb{P}^1 \times \mathbb{P}^1$  under this map will be? Bonus points if you can prove this, but I don't think we currently have the necessary tools.
- (6) Show that if  $\pi$  is an affine morphism, then  $\pi_*$  is exact on quasi-coherent sheaves.
- (7) Show that if  $\pi : X \rightarrow Y$  is a finite morphism between Noetherian schemes, the pushforward of a coherent sheaf is coherent.
- (8) Show that the automorphism group of  $\mathbb{P}^n$  is  $PGL_{n+1} \cong GL_{n+1}/\mathbb{G}_m$ , with the  $\mathbb{G}_m$  being the center of  $GL_{n+1}$ .