Algebraic Geometry II
Homework 6
Due Friday, March 13

(1) Recall that a line bundle L on a nonsingular variety X is very ample if its complete linear system gives a closed embedding into projective space. Show that it is equivalent to require that sections of L separate points and tangent directions, i.e., given two closed points p, $q \in X$, the map $\Gamma(X, L) \to L|_p \oplus L|_q$ is surjective, and the map $\Gamma(X, L) \to L \otimes \mathcal{O}_{X,p}/\mathfrak{m}_p^2$ is surjective.

(2) A line bundle L on a projective variety X is called ample if there is a positive integer n such that $L^{\otimes n}$ is very ample. Show that the following properties are equivalent to ampleness.
   (a) For all sufficiently large n, $L^{\otimes n}$ is very ample.
   (b) For all coherent sheaves $\mathcal{F}$, $\mathcal{F} \otimes L^{\otimes n}$ is globally generated (i.e., is a quotient of a free sheaf) for all sufficiently large n.
   (c) Of the open sets X_f (the locus in X where $f \neq 0$), for $f \in \Gamma(X, L^{\otimes n})$, the affine opens form a basis for the topology of X.

(3) Given an invertible sheaf L and an ample invertible sheaf M, show that there is an n such that $L \otimes M^{\otimes n}$ is very ample. Deduce that every line bundle on a projective variety is the difference between two very ample line bundle.

(4) Show that if L is ample and M is basepoint-free, then $L \otimes M$ is ample.

(5) Show that if L and M are ample then $L \otimes M$ is ample.

(6) Let $\pi : A \to B$ be a morphism. Let $\mathcal{A}$ be a quasicoherent sheaf of algebras on B. Let $X \cong Spec(\mathcal{A})$. Show that $X \times_B A \cong Spec(\pi^* \mathcal{A})$.

(7) Consider the curve $V(y^2 - x^n) \subset \mathbb{A}^2$. If $n \geq 2$, this curve is singular at the origin. How many times must you blow up a point to get a nonsingular curve?