Algebraic Geometry II
Homework 11
Due Monday, April 27

(1) Show that the kernel of a surjective map between flat sheaves is flat.

(2) Show that given an exact sequence of sheaves where all but the first is flat, the first sheaf is flat too.

(3) Let X be a projective scheme with H a very ample divisor on X and $\mathcal{F}$ a coherent sheaf. Show that the function $h_p(m) = \chi(\mathcal{F}(mH))$ is a polynomial. Hint: Use induction on the dimension of X.

(4) Suppose we have a scheme X with a map $\pi : X \to \mathrm{Spec}(k[x]/(x^2))$. Let $\mathcal{F}$ be a quasicoherent sheaf on X. Show that $\mathcal{F}$ is flat over $\mathrm{Spec}(k[x]/(x^2))$ if and only if the natural map $\mathcal{F}/\epsilon\mathcal{F} \to \epsilon\mathcal{F}$ is an isomorphism.

(5) Let C be a smooth curve with $p \in C$. Suppose we have a flat family of closed subschemes of $\mathbb{P}^n$ parametrized by C, i.e., a flat projective morphism $\pi : X \to C \setminus \{p\}$ together with a map $i : X \to \mathbb{P}^n$ such that the natural map $X \to \mathbb{P}^n \times C \setminus \{p\}$. Let $\tilde{X}$ be the scheme-theoretic closure of X in $\mathbb{P}^n \times C$. Show that the morphism $\tilde{X} \to C$ is flat. The fiber over p is called the flat limit of the subschemes X_q, $q \in C \setminus \{p\}$.

(6) Consider the subscheme $V(x^2+y^2+z^2/t) \subset \mathbb{P}^2 \times \mathbb{A}^1_t$. The fibers over points $t \neq 0$ are smooth conics. What is the flat limit of this family over $t = 0$?

(7) Show that if $\pi : X \to Y$ is a proper flat morphism between locally Noetherian reduced schemes with $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, and L_1 and L_2 are invertible sheaves on X which are isomorphic on fibers, then $L_1 \cong L_2 \otimes \pi^* M$, where M is an invertible sheaf on Y.