Algebraic Geometry II
Homework 5
Due Friday, March 6

(1) Let $X \subset \mathbb{A}^3$ be the quadric cone $V(xy - z^2)$. Let $D = V(x, z) \subset X$. Show that $\mathcal{O}_X(D)$ is not an invertible sheaf. Hint: think of the tangent space at the origin.

(2) Show that the pushforward of a quasicoherent sheaf by a qcqs (quasi-compact quasi-separated) morphism is again quasicoherent.

(3) Show that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$. Note that since $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$, we have an example of two smooth projective birational surfaces which are not isomorphic. Hint: Use the fact that the pullback of an invertible sheaf is invertible.

(4) Show that a rational section of a line bundle gives a well-defined Weil divisor.

(5) Consider $\mathbb{P}^1 \times \mathbb{P}^1$ with its two projection maps π_1 and π_2. Consider the invertible sheaf $L = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$. Show that L is very ample. What is the dimension of the complete linear system of L? What do you think the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under this map will be? Bonus points if you can prove this, but I don't think we currently have the necessary tools.

(6) Show that if π is an affine morphism, then π_* is exact on quasi-coherent sheaves.

(7) Show that if $\pi : X \to Y$ is a finite morphism between Noetherian schemes, the pushforward of a coherent sheaf is coherent.

(8) Show that the automorphism group of $\mathbb{P}^n$ is $PGL_{n+1} \cong Gl_{n+1}/\mathbb{G}_m$, with the $\mathbb{G}_m$ being the center of Gl_{n+1}.